An imperfect replacement policy for a periodically tested system with two dependent wear indicators

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ABSTRACT: A periodically tested two-component system is considered, with two possible structures: series and parallel. The accumulating deterioration of both components is modelled with a bivariate non decreasing Lévy process, which takes into account the dependence between the two components. Each component is considered as failed as soon as its univariate deterioration level is beyond a specific threshold. Between inspections, failures remain unrevealed. At inspection times, failed components are instantaneously replaced by new ones (corrective replacements), whereas still working components are left as they are. The repair may hence be imperfect at the system level. To shorten the system down-time, preventive thresholds are next introduced, with a similar replacement policy as for the corrective one otherwise. The system is assessed through cost functions on a finite and infinite horizon times, which are studied with the help of Markov renewal theory. The influence of different parameters (such as the dependence between the two marginal wear indicators) on the cost functions is studied.

Keywords: Reliability; multivariate Lévy processes; dependent wear indicators; Gamma processes; Markov renewal theory.

1 INTRODUCTION

In reliability, stochastic models for deterioration based on actual measurements of the system deterioration level have been the subject of many studies since the last decades. In case of non decreasing deterioration, classical models include compound Poisson processes and Gamma processes, according to whether the deterioration is due to isolated shocks or continuous wear accumulation, see (Abdel-Hameed 1975), (Singpurwalla 1995) or (Van Noortwijk 2009) e.g.. Such classical models both are univariate non decreasing Lévy processes, also called subordinators. We here consider a two-component system, where the deterioration of each component is measured by a univariate subordinator. Because of a common stressing environment, the deterioration levels of the two components are correlated. Hence the need for a bivariate stochastic model with univariate subordinators as margins to describe the system evolution.

Up to our knowledge, multivariate non decreasing wear indicators have not been much studied in the previous literature. Several notable exceptions may however be found such as (**?**) and (Ebrahimi 2004), which both use specific constructions leading to some specific bivariate increasing Lévy processes (though not recognized as such in the quoted papers). As in (Mercier & Pham 2012), we here propose to model the evolution of our two-component system by a general bivariate subordinator (or non decreasing bivariate Lévy process).

Both series and parallel structures are envisionned for the two-component system. Each component is considered as failed as soon at its deterioration level has reached a pre-determined failure threshold. In (Mercier & Pham 2012), the system was assumed to be continously monitored and repairs to be perfect. In the present paper, the deterioration level of the two components is known only through periodic inspections. By an inspection, failed components are instantaneously replaced by new ones (corrective replacements). In case where one single component is down by an inspection, this leads to an imperfect repair at the system level. For the two envisionned structures (series and parallel), the system may remain failed for a while before an inspection. To lower the system down-time, a preventive maintenance policy is considered, where preventive replacements are performed at inspection times, as soon as the deterioration level of a component is observed to be beyond a preventive threshold (lower than the corrective threshold). The preventive maintenance policy is assessed through a cost function, both on a finite and infinite horizon. This cost function takes into account down-time unitary costs, inspections costs as well as replacements costs, with economical dependence between replacement costs. (Simultaneous replacements are less costly than separate replacements). Our model hence takes into account two kinds of dependencies: 1. stochastic dependence between the random deterioration levels of each component (induced by common stresses); 2. economical dependence, which may lead to grouped replacements to lower replacement costs, and consequently implies some kind of functional dependence. Such dependencies (and especially the stochastic one) highly complicate the study, as well as the imperfect repairs.

Similar preventive and corrective threshold-based replacement policies have already been considered in the literature on a large scale in the univariate setting, see (Van Noortwijk 2009) for numerous references in case the system deterioration is modelled by a Gamma process. Papers are much fewer in the multivariate setting. One may however quote (Castanier, Grall, & Berenguer 2005), where the authors envision a two-unit series system with stochastically independent but economically dependent components, in a discrete time setting. Though their study is highly simplified by the assumption of stochastic independence between components, their condition-based inspection scheme is however more complicated than our periodic one. Also, like lots of other papers on similar subjects, they only envision long-time runs whereas we also consider the more difficult case of a finite horizon time, addingly.

The article is organized as follows: in Section 2, the model is presented, both for the unmaintained and preventively maintained system. Section 3 is devoted to theoretical developments whereas Section 4 presents numerical experiments. We finally conclude in Section 5.

2 THE MODEL

2.1 *The unmaintained system*

The deterioration of the two-component system is measured by a bivariate nondecreasing Lévy process $X = \left(X_t^{(1)}, X_t^{(2)}\right)_{t\geq 0}$, also called bivariate subordinator. This means that the process starts from $(0,0)$ and has homogeneous and independent increments, see (**?**) for more details. As in (Mercier & Pham 2012), the process X is assumed to have null drift, so that X is a pure jump process. For sake of simplicity, we also assume that the distribution of X_t admits a density with respect of Lebesgue measure, which is not mandatory for the present study.

For each $i = 1,2$, the i^{th} marginal process $\left(X_t^{(i)}\right)_{t\geq 0}$ stands for the deterioration of the i^{th} component and is an univariate subordinator. The $i-$th component is considered as failed as soon as its deterioration level is beyond threshold L_i. To avoid the trivial case, we assume that $\mathbb{P}\left(X_T^{(1)} > L_1, X_T^{(2)} > L_2\right) > 0$.

The respective probability distribution functions (p.d.f.) of X_t and $X_t^{(i)}$ are denoted by f_t and $f_t^{(i)}$, their cumulative distribution functions (c.d.f.) by F_t and $F_t^{(i)}$, and their survival functions by $\bar{F}_t$ and $\bar{F}_t^{(i)}$.

The state of the system is perfectly controlled via periodic inspections at time $0, T, 2T, \ldots$ Between inspections, failures remain unrevealed. At time nT, $n \geq 0$, only failed components are replaced (corrective replacements). Replacements are assumed to be instantaneous and perfect. This means that, by an inspection, deterioration levels of failed components are reset to zero whereas they are left unchanged otherwise.

2.2 *The preventive maintenance policy*

In order to avoid failures and to shorten down periods, preventive maintenance thresholds M_i are next introduced (with $0 \leq M_i \leq L_i$, $i = 1,2$), with a similar replacement policy as for corrective replacements otherwise. More specifically, at time nT, $n \geq 0$, if the deterioration level of the i^{th} component is between M_i and L_i , a preventive replacement is performed. If its deterioration is beyond L_i, the component is failed and a corrective replacement takes place. Preventive replacements (PR) are assumed to be instantaneous and perfect, just as for corrective replacements (CR).

This preventive maintenance (PM) policy is illustrated in Figure 1, where, at time T, the second component is preventively replaced because its deterioration level exceeds M_2 but remains below L_2, while the first component is left as it is ($X_T^{(1)} < M_1$). At time $2T$, the first component is preventively replaced while the second component is left as it is. At time $3T$, both components are left as they are. At time $4T$, a simultaneous replacement takes place: the first component is preventively replaced whereas a corrective replacement is performed on the second one. Components are next left as they are at time $5T$ and one single corrective replacement takes place at time $6T$ for the second component. In all this sequence, only one complete replacement is performed at the system level, at time $4T$. All the other maintenance actions are imperfect (at the system level).

This replacement policy does not depend on the system structure (series or parallel): it is the same for both structures. However, the system state (up or down) depends on its structure, as well as the down-time duration.

Taking $M_i = L_i$ for $i = 1,2$, the unmaintained system appears as a special case of the preventively maintained system. Taking $M_i = 0$ for $i = 1,2$, the system is replaced every T time units. The classical

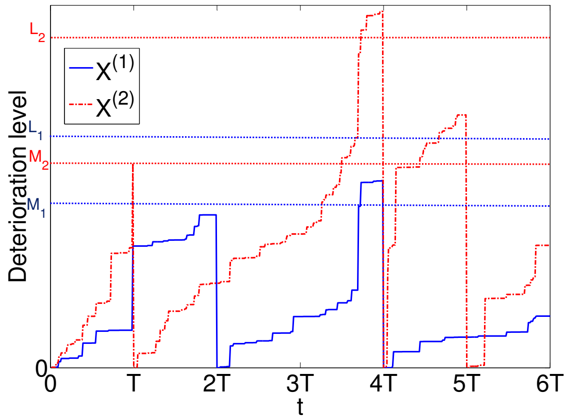

Figure 1: The preventive maintenance policy

periodic replacement policy with no repair at failure and period T then appears as a special case of the PM policy.

To assess the PM policy, cost functions are considered, which takes into account a down-time unitary cost per unit time (c_u), inspection costs (c_p) and replacement costs. The cost of simultaneous replacement of both components is $c_1 + c_2 + c_r$. If only the i^{th} component is replaced ($i = 1, 2$), the cost is $c_i + c_r$. This induces an economical dependence between cost replacements.

3 THEORETICAL RESULTS

3.1 *Structure of the stochastic process*

Let $Y = \left(Y_t^{(1)}, Y_t^{(2)}\right)_{t\geq 0}$ be the stochastic process describing the maintained system. Each time both components are simultaneously replaced, the system is as good as new and its future evolution is stochastically identical to taht of the initial system and independent of its past. So, on one side, the stochastic process $Y = (Y_t)_{t\geq 0}$ appears as a regenerative process, where the simultaneous replacements of both components are the regeneration times, with generic length τT, where

$$\tau = \inf\left(n \geq 1 : Y_{nT} = (0,0)\right). \tag{1}$$

This property is used in Subsection 3.3.

On the other side, considering the system state after each inspection, the sequence $(Y_{nT})_{n\geq 0}$ is a Markov chain with continuous state space $[0, M_1] \times [0, M_2]$. Indeed, regardless of whether the components are replaced or not at inspection time nT, their future evolution after time nT only depends on their state at time nT. This means that $(Y_t)_{t\geq 0}$ is a semi-regenerative process, with $(Y_{nT})_{n\geq 0}$ as embedded Markov chain. Both processes $\left(Y_{nT}^{(i)}\right)$, $i = 1, 2$ have the same property and also are semi-regenerative processes, with $(Y_{nT}^{(i)})_{n\geq 0}$, $i = 1, 2$ as embedded Markov chains and state space $[0, M_i]$.

We next provide the transition kernel of the different Markov chains, namely $\mathcal{Q}(x, dy)$ and $Q^{(i)}(x_i, dy_i)$, $i = 1, 2$, with

$$\mathcal{Q}(x, dy) = \mathbb{P}\left(Y_T \in dy | Y_0 = x\right) = \mathbb{P}_x\left(Y_T \in dy\right)$$

for all $x = (x_1, x_2) \in [0, M_1] \times [0, M_2]$, $dy = (dy_1, dy_2)$ and

$$\begin{aligned} Q^{(i)}(x_i, dy_i) &= \mathbb{P}\left(Y_T^{(i)} \in dy_i | Y_0^{(i)} = x_i\right) \\ &= \mathbb{P}_{x_i}\left(Y_T^{(i)} \in dy_i\right) \end{aligned}$$

for all $x^{(i)} \in [0, M_i]$, $i = 1, 2$.

For $i = 1, 2$, there are two possible scenarios for the i^{th} component at atime T: either the component is replaced by a new one and its level deterioration is reset to 0, or it is left as it is. Starting from x_i, the first scenario happens with the probability

$$\mathbb{P}_{x_i}\left(X_T^{(i)} > M_i\right) = \overline{F}_T^{(i)}(M_i - x_i).$$

As for the second scenario, it means that the level of the i^{th} component at time T is $x_i + X_T^{(i)}$, with $x_i + X_T^{(i)} \leq M_i$. This provides the following transition kernel:

$$\begin{aligned} Q^{(i)}(x_i, dy_i) = &\overline{F}_T^{(i)}(M_i - x_i)\delta_0(dy_i) \\ &+ 1_{[x_i, M_i]}(y_i) f_T^{(i)}(y_i - x_i) dy_i \end{aligned}$$

where $x_i \in [0, M_i]$ and where $\delta_0(dy_i)$ stands for the Dirac measure at 0.

As for the whole system, at time T, there are three possibilities: either no replacement, or one single replacement, or two simultaneous replacements. According to which component is replaced in case of one single replacement (component 1 or 2), this leads to four different possible scenarios and provides the following transition kernel:

$$\mathcal{Q}(x, dy) = \sum_{i=1}^{4} \mathcal{Q}_i(x, dy)$$

with

$$\mathcal{Q}_1(x,dy) = \mathbf{1}_{[x_1,M_1]}(y_1)\,\mathbf{1}_{[x_2,M_2]}(y_2) \times f_T(y_1-x_1,y_2-x_2)dy_1\,dy_2, \quad (2)$$

$$\mathcal{Q}_2(x,dy) = \mathbf{1}_{[x_2,M_2]}(y_2) \times \left(\int_{M_1}^{+\infty} f_T(u_1-x_1,y_2-x_2)du_1\right)\delta_0(dy_1)\,dy_2, \quad (3)$$

$$\mathcal{Q}_3(x,dy) = \mathbf{1}_{[x_1,M_1]}(y_1) \times \left(\int_{M_2}^{+\infty} f_T(y_1-x_1,u_2-x_2)du_2\right)dy_1\,\delta_0(dy_2), \quad (4)$$

$$\mathcal{Q}_4(x,dy) = \overline{F}_{X_T}(M_1-x_1,M_2-x_2) \times\delta_0(dy_1)\delta_0(dy_2) \quad (5)$$

for $x=(x_1,x_2)\in[0,M_1]\times[0,M_2]$ and $y=(y_1,y_2)\in[0,M_1]\times[0,M_2]$.

3.2 *The cost function on a finite horizon time*

The total cost depends on the probability of replacement of one or two components at inspection times, and of the mean down-time duration. For $x=(x_1,x_2)\in[0,M_1]\times[0,M_2]$, we hence introduce $U(x,[0,t[)$ to be the mean down-time duration on $[0,t[$ given that the system starts from $X_0=x$. We also set $R_{12}(x,nT)$ and $R_i(x,nT)$ to be the probabilities of replacement of both components and of the i^{th} component ($i=1,2$) at time nT, respectively.

Then, given that $X_0=x$, one easily gets that the mean cost on $[0,t[$ is

$$C(x,[0,t[)$$

$$= c_u U(x,[0,t[) + (c_1+c_r)\sum_{n:nT<t} R_1(x_1,nT)$$

$$+ (c_2+c_r)\sum_{n:nT<t} R_2(x_2,nT) - c_r\sum_{n:nT<t} R_{12}(x,nT)$$

$$+ c_p\lfloor\frac{t}{T}\rfloor,$$

where $\lfloor\frac{t}{T}\rfloor$ stands for the integer part of $\frac{t}{T}$.

Note that the cost is computed on $[0,t[$ so that, in case $t=nT$, $n\geq 1$, no replacement cost is considered at time t. (Indeed, we do not replace components once the horizon time t is reached). Of course, this does not inferer on the mean down time: $U(x,[0,t[) = U(x,[0,t]) = U(x,]0,t])$.

We now have to compute the different quantities involved in $C(x,[0,t[)$. Given that $X_0=(x_1,x_2)\in[0,M_1]\times[0,M_2]$, it is easy to see that at the first inspection time T:

$$R_i(x,T) = \overline{F}_T^{(i)}(M_i-x_i),$$

$$R_{12}(x,T) = \overline{F}_T(M_1-x_1,M_2-x_2).$$

As for the mean down time on $[0,t[$, it depends on the system structure. For $t\leq T$, we get:

$$U(x,[0,t[) = \int_0^T \overline{F}_u(L_1-x_1,L_2-x_2)\,du$$

for a parallel system and

$$U(x,[0,t[) = t - \int_0^t F_u(L_1-x_1,L_2-x_2)\,du$$

for a series system.

From the second period, conditionning on Y_T and using the Markov property at time T, we get for all $n\geq 2$:

$$\begin{aligned} R_{12}(x,nT) &= \mathbb{E}_x\left[\mathbf{1}_{\left\{Y_{nT}^{(1)}>M_1,Y_{nT}^{(2)}>M_2\right\}}\right] \\ &= \mathbb{E}_x\left[\mathbb{E}_x\left(\mathbf{1}_{\left\{Y_{nT}^{(1)}>M_1,Y_{nT}^{(2)}>M_2\right\}}|Y_T\right)\right] \\ &= \mathbb{E}_x\left[R_{12}(Y_T,(n-1)T)\right] \\ &= \iint_{\mathbb{R}_+^2} R_{12}(y,(n-1)T)\mathcal{Q}(x,dy). \end{aligned}$$

Using similar arguments for the other quantities, we get the following Markov renewal equations (MRE).

Proposition 1 *For $n\geq 2$, we have:*

$$R_i(x_i,nT) = \int_{\mathbb{R}_+} R_i(y_i,(n-1)T)Q^{(i)}(x_i,dy_i)$$

for $i=1,2$ and $x_i\in[0,M_i]$ and

$$R_{12}(x,nT) = \iint_{\mathbb{R}_+^2} R_{12}(y,(n-1)T)\mathcal{Q}(x,dy),$$

$$U(x,[T,t[) = \iint_{\mathbb{R}_+^2} U(y,[0,t-T[)\mathcal{Q}(x,dy)$$

for $x=(x_1,x_2)\in[0,M_1]\times[0,M_2]$ and $t>T$.

Such results allow to recursively compute the different quantities involved in the cost.

3.3 *Cost on infinite horizon*

Recalling that τT is the length of a generic cycle for the process $(Y_t)_{t\geq 0}$ seen as regenerative process, see (1), one can check that $\mathbb{E}_x(\tau)$ is finite, for any starting point x. The embedded Markov chain $(Y_{nT})_{n\geq 0}$ hence is a Harris chain and it admits a unique stationary distribution π (say). This distribution is the single solution of the integral equation $\pi\mathcal{Q}=\pi$. Substituting the kernel $\mathcal{Q}$ by its four-part expression $(2-5)$, we get the following result by identifying the terms with respect to $dx_1\,dx_2$, $\delta_0(dx_1)\,dx_2$, $dx_1\,\delta_0(dx_2)$ and $\delta_{(0,0)}(dx_1,dx_2)$.

Proposition 2 *Let:*

$$\begin{aligned} g(y_1,y_2) &= \int_{M_1}^{\infty} f_T(u_1-y_1,y_2)\,du_1, \\ h(y_1,y_2) &= \int_{M_2}^{\infty} f_T(y_1,u_2-y_2)\,du_2, \\ k(y_1,y_2) &= \overline{F}_T(M_1-y_1,M_2-y_2) \end{aligned}$$

for all $(y_1,y_2)\in[0,M_1]\times[0,M_2]$. *Then, the invariant distribution* π *of the Markov chain* $(Y_{nT})_{n\geq 0}$ *is of the shape:*

$$\pi(dx) = a_{12}(x)\,dx + a_2(x_2)\,\delta_0(dx_1)\,dx_2 + a_1(x_1)\,dx_1\,\delta_0(dx_2) + a_4\,\delta_{(0,0)}(dx)$$

where $x=(x_1,x_2)$ *and where* $a_{12}(x)$, $a_2(x_2)$, $a_1(x_1)$, a_4 *are the single solution of the following set of integral equations:*

$$\begin{aligned} a_{12}(x) = &\iint_{[0,M_1]\times[0,M_2]} a_{12}(y) f_T(x-y)\,dy \\ &+ \int_0^{M_2} a_2(y_2) f_T(x_1,x_2-y_2)\,dy_2 \\ &+ \int_0^{M_1} a_1(y_1) f_T(x_1-y_1,x_2)\,dy_1 + a_4\, f_T(x), \end{aligned}$$

$$\begin{aligned} a_1(x_1) = &\iint_{[0,M_1]\times[0,M_2]} a_{12}(y) h(x_1-y_1,y_2)\,dy \\ &+ \int_0^{M_2} a_2(y_2) h(x_1,y_2)\,dy_2 \\ &+ \int_0^{M_1} a_1(y_1) h(x_1-y_1,0)\,dy_1 + a_4\, h(x_1,0), \end{aligned}$$

$$\begin{aligned} a_2(x_2) = &\iint_{[0,M_1]\times[0,M_2]} a_{12}(y) g(y_1,x_2-y_2)\,dy \\ &+ \int_0^{M_2} a_2(y_2) g(0,x_2-y_2)\,dy_2 \\ &+ \int_0^{M_1} a_1(y_1) g(y_1,x_2)\,dy_1 + a_4\, g(0,x_2), \end{aligned}$$

$$\begin{aligned} a_4 = &\iint_{[0,M_1]\times[0,M_2]} a_{12}(y) k(y)\,dy \\ &+ \int_0^{M_2} a_2(y_2) k(0,y_2)\,dy_2 \\ &+ \int_0^{M_1} a_1(y_1) k(y_1,0)\,dy_1 + a_4 k(0,0), \end{aligned}$$

for all $x=(x_1,x_2)\in[0,M_1]\times[0,M_2]$.

These equations are numerically solved using an iterative method.

Now, let $c(I)$ be the random cost on any interval $I\subset\mathbb{R}_+$. As $(Y_t)_{t\geq 0}$ is a regenerative process with finite mean length cycle $\mathbb{E}_0(\tau T)=T\mathbb{E}_0(\tau)$, we know from classical renewal theory (Asmussen 2003) that the asymptotic unitary cost

$$C_\infty = \lim_{t\to\infty} \frac{c([0,t[)}{t}$$

exists (almost surely) and that it may be computed via:

$$C_\infty = \frac{\mathbb{E}_0(c(\tau T))}{\mathbb{E}_0(\tau T)} = \frac{\mathbb{E}_0(c(\tau T))}{T\mathbb{E}_0(\tau)}.$$

Furthermore, according to (Cocozza-Thivent), we have

$$\mathbb{E}_0(c(\tau T)) = \mathbb{E}_0(\tau)\mathbb{E}_\pi\left[C(\cdot,]0,T])\right]$$

where π is the stationary distribution of $(Y_{nT})_{n\geq 1}$ provided by Proposition 2 and where

$$\mathbb{E}_\pi\left[C(\cdot,]0,T])\right] = \iint_{[0,M_1]\times[0,M_2]} C(x,]0,T])\pi(dx).$$

So we get

$$C_\infty = \frac{\mathbb{E}_\pi\left[C(\cdot,]0,T])\right]}{T}.$$

Based on the previous results for the mean cost on a finite horizon and for the invariant distribution π, we are now able to compute the asymptotic unitary cost C_∞.

4 NUMERICAL EXPERIMENTS

4.1 *Bivariate Gamma process*

A specific model similar to (Mercier & Pham 2012) is here used, which we call bivariate Gamma process. We recall its construction, for sake of completeness. We first remind that a univariate Gamma process with parameters (a,b) (where $a,b>0$) is a subordinator Z such that for every $t\geq 0$, the random variable Z_t is Gamma distributed $\Gamma(at,b)$ with p.d.f.:

$$f_{at,b}(x) = \frac{1}{\Gamma(at)} b^{at} e^{-bx} x^{at-1} 1_{\{x>0\}}.$$

We only envision the case $b=1$ in the following, which is no restriction.

Starting from three independent univariate Gamma processes $\left(Z_t^{(i)}\right)_{t\geq 0}$ with parameters $(\alpha_i,1)$ for $i=1,2,3$ (where $\alpha_1,\alpha_2,\alpha_3>0$), we set

$$X_t^{(1)} = Z_t^{(1)} + Z_t^{(3)} \text{ and } X_t^{(2)} = Z_t^{(2)} + Z_t^{(3)}.$$

The process $(X_t)_{t\geq 0} = \left(X_t^{(1)},X_t^{(2)}\right)_{t\geq 0}$ then is a bivariate subordinator with Gamma marginal processes and marginal parameters $(a_i,1)$ where $a_i=\alpha_i+\alpha_3$ for $i=1,2$. Pearson's correlation coefficient between the two random variables $X_t^{(1)}$ and $X_t^{(2)}$ is independent of t and given by

$$\rho = \frac{\alpha_3}{\sqrt{a_1 a_2}}. \tag{6}$$

We consequently have $\alpha_1 = a_1-\rho\sqrt{a_1a_2}$, $\alpha_2=a_2-\rho\sqrt{a_1a_2}$ and $\alpha_3=\rho\sqrt{a_1a_2}$, with $0\leq\rho\leq\rho_{\max}=\min\left(\sqrt{\frac{a_1}{a_2}},\sqrt{\frac{a_2}{a_1}}\right)$. Two equivalent alternate parameterizations hence are available for $(X_t)_{t\geq 0}$: either $(\alpha_1,\alpha_2,\alpha_3)$ or (a_1,a_2,ρ). Besides, all the dependence between the marginal processes is contained in the linear correlation coefficient ρ.

Table 1: Examples parameters

	Structure	a_1	a_2	ρ	L_1	L_2	M_1	M_2	T	c_1	c_2	c_r	c_p	c_u	t_0
Ex .1	-	4	9	0.5	-	-	0.6	0.9	0.5	-	-	-	-	-	-
Ex.2	-	4	4	0.6	-	-	0.5	0.3	0.6	-	-	-	-	-	
Ex. 3	Parallel	4	4	0.5	0.7	0.6	0.5	0.5	-	1	1	1	-	10	4
Ex. 4	Series	4	9	0.5	1.2	1.4	-	-	0.5	0	-	-	0	-	4T
Ex. 5	-	4	9	-	-	-	4	5	1	-	-	-	-	-	4T

4.2 *Examples*

Parameters for all examples are displayed in Table 1, as well as the system structure when necessary (computations of mean down times and costs).

Example 1 *Two parts of the invariant distribution π of $(Y_{nT})_{n\geq 0}$ are displayed in Figure 2: functions $a_1(x_1)$ and $a_{12}(x)$, which are the p.d.f.'s of π with respect of $dx_1\ \delta_0(dx_2)$ (replacement of component 2 only) and $dx_1\ dx_2$ (no replacement), respectively. With the chosen parameters, we observe that $a_1(x_1)$ is increasing with x_1 and that $a_{12}(x)$ is concave. Also, we get $a_4 \simeq 0.89$, so that the probability of simultaneous replacement of both components is high for large times.*

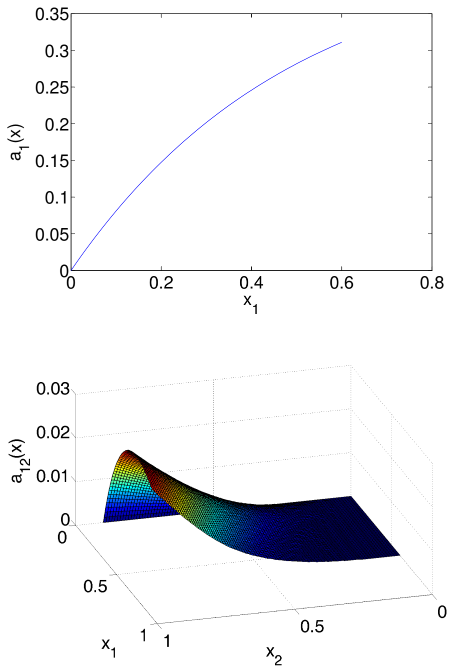

Figure 2: Parts of the invariant distribution of $(Y_{nT})_{n\geq 0}$, Example 1

Example 2 *The convergence of the distribution of $(Y_{nT})_{n\geq 0}$ towards π, which has been theoretically proved, is illustrated in Figure 3 through the numerical convergence of the mean rate of simultaneous replacements on $[0, nT]$, that is of $\frac{1}{nT}\sum_{i=1}^{n} R_{12}(0, iT)$ towards the asymptotic rate $\frac{1}{T}\mathbb{E}_\pi(R_{12}(\cdot, T)) = \frac{1}{T}\int_{\mathbb{R}_+^2} R_{12}(x, T)\,\pi(dx)$.*

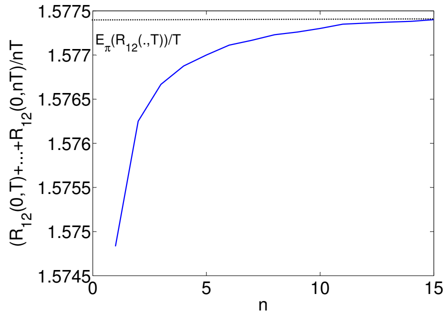

Figure 3: Illustration of the convergence of $(Y_{nT})_{n\geq 0}$, Ex. 2

Example 3 *This example illustrates the variations of the mean rate of simultaneous replacements (asymptotic case) and of the cumulated mean number of simultaneous replacements on $[0, t_0[$ (finite time case) with respect of the period T (see Figure 4). For the finite horizon case $[0, t_0[$, this mean number of replacements is not continuous with respect of T. The discontinuity points are the points T such that there exists some number n satisfying $t_0 = nT$, because at these points, we do not consider possible replacements at t_0. As for the asymptotic case, the rate of simultaneous replacements seems continuous with respect of T.*

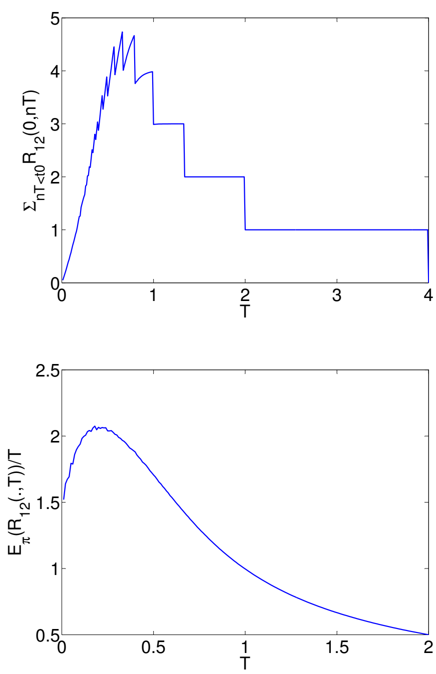

Figure 4: Mean rate (asymptotic case, lower figure) and mean number (finite time case, upper figure) of simultaneous replacements as a function of T, Example 3

Figure 5 (upper) shows that, as expected, the cumulated mean down time on $[0, t_0[$ is increasing with T. Figure 5 (lower) shows that the asymptotic unavailability $\mathbb{E}_\pi \left(U\left([0, T]\right)\right)/T$ also is increasing with T.

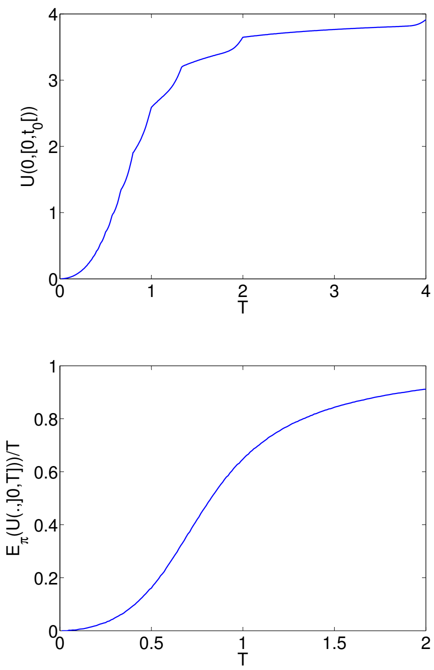

Figure 5: Mean down time in the finite horizon case (upper figure) and asymptotic unavailability (lower figure) as a function of T, Example 3

The asymptotic unitary cost is plotted in Figure 6 as a function of T. The function is convexe and admits a single minimum at $T_{opt} \simeq 0.52$.

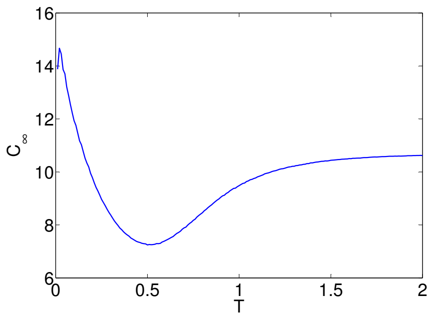

Figure 6: The asymptotic unitary cost as a function of T, Example 3, $c_p = 0.01$

Example 4 *The influence of the preventive maintenance thresholds (M_1, M_2) on the cost functions of a series system is here studied. We take $c_p = c_1 = 0$ and consider several cases for (c_2, c_u, c_r). As we can see, both on a finite and infinite horizons, the cost function may be convex (see Figure 7), concave (see Figure 8) or more complicated (see Figure 9).*

So it is hard to say anything about the shape of the cost functions with respect to the preventive maintenance thresholds (M_1, M_2). This is due to the fact that the mean number of replacements and the down-time durations may have reverse concavity with respect of (M_1, M_2).

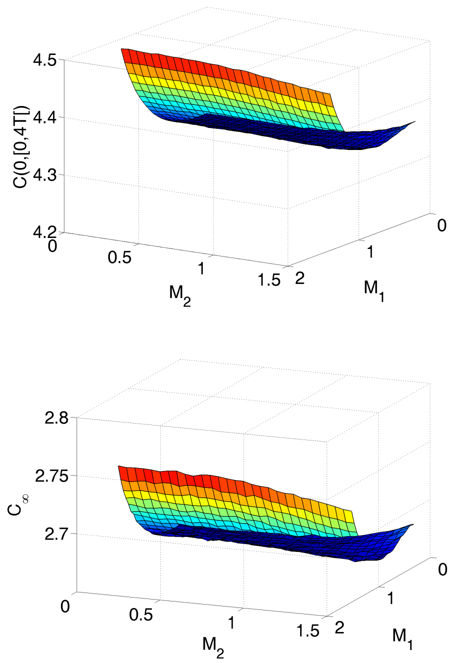

Figure 7: The cost functions of (M_1, M_2), Example 4, $c_2 = 0$, $c_r = 1$, $c_u = 2$

Example 5 *We finally look at the influence of the dependence between the two components on criterion of interest. Let us recall that, for the bivariate Gamma model, this dependence is measured by Pearson's correlation ρ, see (6). The probability of simultaneous replacements of both components is plotted in Figure 10. We can see that it is not monotone with respect of ρ, so that costs will not be monotone with respect of the dependence either, neither for the finite nor for the infinite horizon time.*

5 CONCLUSIONS

We here proposed a preventive maintenance policy for a periodically tested system modeled by a bivariate subordinator. The PM policy is assessed through a cost function on both finite and infinite horizon time. Influences of different parameters (preventive maintenance thresholds, the duration between two inspections and the dependence between components) are studied from a numerical point of view. Most of the time, the variations of the cost functions are very complicated with respect to each parameter, so that we cannot expect to get theoretical results on such points. However, some points might be studied more carefully. For instance, we have observed on a few cases that the asymptotic unavailability was increasing with respect to the duration between inspections. Is it possible to prove it theoretically? This is an open question.

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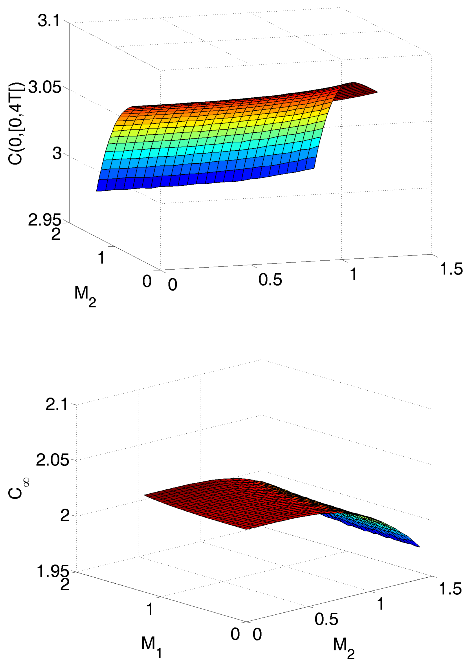

Figure 8: The cost functions of (M_1, M_2), Example 4, $c_2 = 1$, $c_r = 0$, $c_u = 0.1$

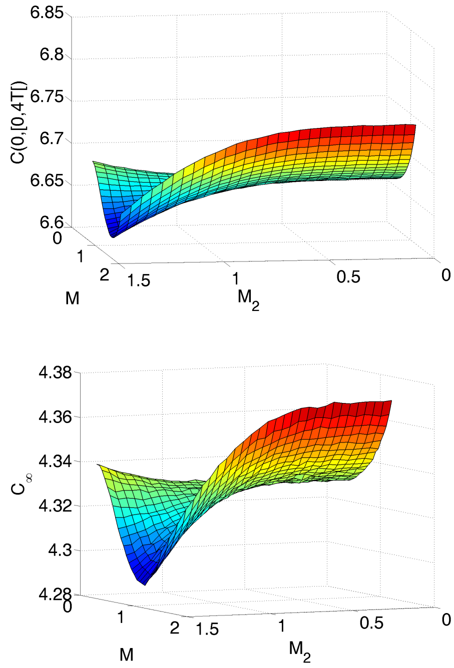

Figure 9: The cost functions of (M_1, M_2), Example 4, $c_2 = 0$, $c_r = 2$, $c_u = 1$

work. This work was also supported for both authors by the French National Research Agency (ANR), AMMSI project, ref. ANR 2011 BS01-021.

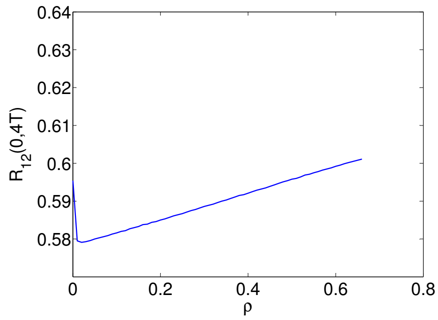

Figure 10: Probability of replacement of two components with respect to ρ, Example 5

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